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Note

Two classes of q-ary codes based on group divisible association schemes

Ying Miao[a,*], Sanpei Kageyama[b]

[a]Institute of Policy and Planning Sci., University of Tsukuba, 1-1-1 Tennodai, Tsukuba 305-8573, Ibaraki, Japan

[b]Department of Mathematics, Hiroshima University, Higashi-Hiroshima 739-8524, Japan

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Abstract

By means of Hall's Marriage Theorem, a class of q-ary codes are obtained through block designs based on group divisible association schemes with two associate classes. If these block designs are further semiframes, another class of q-ary codes are also constructed. © 1999 Elsevier Science B.V. All rights reserved

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1. Introduction

As is well known, certain combinatorial structures have played a significant role in coding theory. Several results from design theory could be used to construct 'good' codes. Such examples include Semikov and Zinoviev's construction [10] for a class of optimal equidistant q-ary codes through resolvable balanced incomplete block designs, and Sinha's construction [11] for a class of q-ary codes through nested balanced incomplete block designs. Here, as usual, a q-ary code C of length n means a subset $C \subseteq F_q^n$ of the set of all n-tuples with components from an alphabet F_q of size q.

In this paper, we will exploit the structures of block designs based on group divisible association schemes with two associate classes to provide q-ary codes by means of Hall's Marriage Theorem. If these block designs are further semiframes, they are then utilized to give another class of q-ary codes.

We adapt some definitions here for the convenience of the reader. For those undefined terms, the reader is referred to [2–4, 13].

* Corresponding author. E-mail: miao@sk.tsukuba.ac.jp.

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An association scheme with f associate classes on a set $\mathscr{V}$ of v elements is a family of f symmetric binary relations on $\mathscr{V}$ such that

(1) any two distinct elements of $\mathscr{V}$ are ith associates for exactly one value of i, where $1 \leqslant i \leqslant f$;

(2) each element of $\mathscr{V}$ has n_i ith associates, $1 \leqslant i \leqslant f$;

(3) for each i, $1 \leqslant i \leqslant f$, if x and y are ith associates, then there are p^i_{jl} $(=p^i_{lj})$ elements of $\mathscr{V}$ which are both jth associates of x and lth associates of y.

A group divisible association scheme, based on a set $\mathscr{V}$ of $v = fg$ elements divided into f groups $\mathscr{G}$ of g elements each, is an association scheme with two associate classes such that any two elements in the same group are first associates while those in different groups are second associates.

A group divisible design (GDD) $(\mathscr{V}, \mathscr{G}, \mathscr{B})$ with parameters $v, b, r, k, \lambda_1, \lambda_2$ is a block design, based on a group divisible association scheme on $\mathscr{V}$, with a collection $\mathscr{B}$ of b blocks each of size k and with replication number r, such that if two distinct elements x and y are ith associates, $i = 1, 2$, they occur together in exactly λ_i blocks. When $\lambda_1 = 0$, if the collection of blocks $\mathscr{B}$ can be written as a disjoint union $\mathscr{B} = \mathscr{P} \cup \mathscr{Q}$, where $\mathscr{P}$ is partitioned into parallel classes, each of which partitions $\mathscr{V}$, and $\mathscr{Q}$ is partitioned into partial parallel classes, each of which partitions $\mathscr{V} - G$ for some group $G \in \mathscr{G}$, then the GDD is called a semiframe, denoted by (k, λ_2)-semiframe of type g^f.

2. An application of Hall's Marriage Theorem to q-ary codes

Given a finite set $\mathscr{V}$ and its m non-empty subsets $S_1, \ldots, S_m$, $(v_1, \ldots, v_m)$ is called a system of distinct representatives (SDR) for $S_1, \ldots, S_m$, if

(1) $v_i \in S_i$ for $i = 1, 2, \ldots, m$; and

(2) $v_i \neq v_j$ for $i \neq j$, $i, j = 1, 2, \ldots, m$.

For any set $J \subseteq \{1, \ldots, m\}$ of indices, define $S(J) = \bigcup_{j \in J} S_j$. Then we have the following Hall's Marriage Theorem.

Theorem 2.1 (Hall [5]). *A necessary and sufficient condition for $S_1, \ldots, S_m$ to possess an SDR is that*

$$|S(J)| \geqslant |J| \quad \textit{for all } J \subseteq \{1, \ldots, m\}.$$

This theorem was originally due to Hall [5] but now there are many different proofs and many generalizations. They are closely connected with the theory of flows in networks, and have applications in constructing various types of combinatorial structures, such as Latin rectangles, Youden squares, block designs in which the elements can be rearranged within blocks such that every element occurs as equally often as possible in every row, etc; see, e.g., [1,4,7–9,12]. Here we provide a new application of this theorem to give some q-ary codes, where, as usual, an $(n, M, d; q)$ code denotes a q-ary code with length n, size M, and minimum (Hamming) distance d.

Theorem 2.2. *Let $r = sk + t$, $s \geqslant 0$, $0 \leqslant t \leqslant k-1$. Then the existence of a GDD with parameters $v, b, r, k, \lambda_1, \lambda_2$ implies the existence of the following codes:*

(1) *a $(b, v, 2r - \max\{\lambda_1, \lambda_2\};\ k+1)$ code of constant weight r where for any two distinct elements x and y, $d(x, y) = 2r - \lambda_i$ if x and y are ith associates, $i = 1, 2$, and for any symbol θ, $1 \leqslant \theta \leqslant k$, θ occurs s or $s+1$ times in each codeword, and symbol 0 occurs $b - r$ times in each codeword; in particular if $t = 0$, then for every symbol θ, $1 \leqslant \theta \leqslant k$, θ occurs s times in each codeword;*

(2) *a $(b, v+k,\ \min\{2r - \max\{\lambda_1, \lambda_2\},\ b-s-1\};\ k+1)$ code; in particular if $t = 0$, a $(b, v+k,\ \min\{2r - \max\{\lambda_1, \lambda_2\},\ b-s\};\ k+1)$ code; and*

(3) *a $(b, v+k+1,\ \min\{2r - \max\{\lambda_1, \lambda_2\},\ b-s-1,\ r\};\ k+1)$ code; in particular if $t = 0$, a $(b, v+k+1, \min\{2r - \max\{\lambda_1, \lambda_2\},\ b-s,\ r\};\ k+1)$ code.*

Proof. Let the GDD be based on $\{1, 2, \ldots, v\}$. Since $bk = rv = svk + tv$, k divides tv. Let $u = v - tv/k$. Add $k - t$ new replications of element x in u dummy columns to the incidence matrix of the GDD I to get a new matrix I', which is a $v \times (b+u)$ $(0,1)$-matrix, having k 1's in each column and $r' = (s+1)k$ 1's in each row. For each element x, $1 \leqslant x \leqslant v$, take the following procedure. Choose k replications of element x from I' and replace them by symbol $(x, 1)$. Choose other k replications of element x and replace them by symbol $(x, 2)$. Iterate this procedure until all the replications of element x have been replaced by $(x, 1), \ldots, (x, s+1)$. The only restriction in this procedure is that the $k - t$ new replications of element x in the dummy columns are all replaced by the same symbol, say $(x, s+1)$, which is always possible since $k - t \leqslant k$.

Clearly, the resulting matrix I'' is a $v \times (b+u)$ matrix with entries 0 and (x, j), $1 \leqslant x \leqslant v$, $1 \leqslant j \leqslant s+1$, having k entries (x, j) in each row for each x and j, and having k non-zero symbols in each column. Define $S_l = \{(x, j)\colon (x, j) \in l\text{th column of } I''\}$, $1 \leqslant l \leqslant b+u$. Applying Theorem 2.1 to these S_l's, we get an SDR, say $r_1 = (r_{1,1}, \ldots, r_{1,b+u})$. Again, apply Theorem 2.1 to $(S_l - \{r_{1,l}\})$'s to get another SDR, say $r_2 = (r_{2,1}, \ldots, r_{2,b+u})$. Iterate Theorem 2.1 until we get k SDR's, say $r_1, \ldots, r_k$. Note that each r_θ contains each symbol (x, j) once, $1 \leqslant x \leqslant v$, $1 \leqslant j \leqslant s+1$. Furthermore, for each x, symbol $(x, s+1)$ can appear at most once in each r_θ, and this appearance will either be in the dummy columns or the original columns.

Now delete all the dummy columns and transform symbols (x, j) in r_θ to θ, $1 \leqslant x \leqslant v$, $1 \leqslant j \leqslant s+1$, $1 \leqslant \theta \leqslant k$. Denote the resulting new matrix by I'''. Taking the rows of I''' as codewords, we get the code (1). Add k codewords $(\theta, \ldots, \theta)$ of size b to this code to obtain (2), and further addition of a codeword $(0, \ldots, 0)$ of size b to (2) yields (3).

When $t = 0$, this code can be constructed without adding any dummy column so that for any symbol θ, $1 \leqslant \theta \leqslant k$, θ occurs s times in each codeword, and symbol 0 occurs $b - r$ times in each codeword. $\square$

Note that a balanced incomplete block design (BIBD) with parameters v, b, r, k, λ can be regarded as a GDD with parameters $v = fg, b, r, k$, $\lambda_1 = 0$, $\lambda_2 = \lambda$ for $g = 1$ or $f = v$. This is sometimes denoted by $B(k, \lambda; v)$ since the other two parameters are not independent of these three ones.

Corollary 2.3. *Let $r = sk + t$, $s \geqslant 0$, $0 \leqslant t \leqslant k-1$. Then the existence of a BIBD with parameters v, b, r, k, λ implies the existence of the following codes*:

(1) *an equidistant $(b, v, 2r-\lambda;\ k+1)$ code of constant weight r where for any symbol θ, $1 \leqslant \theta \leqslant k$, θ occurs s or $s+1$ times in each codeword, and element 0 occurs $b-r$ times in each codeword; in particular if $t=0$, then for every symbol θ, $1 \leqslant \theta \leqslant k$, θ occurs s times in each codeword; and*

(2) *a $(b, v+k,\ \min\{2r-\lambda,\ b-s-1\};\ k+1)$ code; in particular if $t=0$, a $(b, v+k,\ \min\{2r-\lambda,\ b-s\};\ k+1)$ code; and*

(3) *a $(b, v+k+1,\ \min\{r,\ b-s-1\};\ k+1)$ code; in particular if $t=0$, a $(b, v+k+1,\ \min\{r,\ b-s\};\ k+1)$ code.*

Note that the code (3) in Corollary 2.3 is also equidistant when $t=0$ provided that $r=b-s$, which means that $v=k+1, b=sv$, $r=sk$, $\lambda=s(k-1)$ for some positive integer $s \geqslant 1$.

Corollary 2.4. *Let $s \geqslant 1$ be an integer. Then the existence of a $B(k, s(k-1);\ k+1)$ implies the existence of an equidistant $(s(k+1), 2(k+1), sk; k+1)$ code.*

Some codes obtained by this method are fairly 'good' as the following shows.

Theorem 2.5 (Tonchev [13], Theorem 2.3.5). *For any equidistant $(n, M, d; q)$ code, $d \leqslant nM(q-1)/\{(M-1)q\}(=d_{\mathrm{opt}}$, say), where the equality is achieved if and only if M is a multiple of q and each of the symbols $0, 1, \ldots, q-1$ occurs exactly M/q times in each column of the $M \times n$ matrix formed by the codewords.*

An equidistant code that achieves the equality in Theorem 2.5 is said to be optimal. A necessary condition for the existence of an optimal equidistant code is that d_{opt} be an integer. If d_{opt} is not an integer, i.e., the equidistant code is not optimal, then the code with $d = \lfloor d_{\mathrm{opt}} \rfloor$ is called nearly optimal, which is obviously the best possible equidistant code.

Now we compare $d_0 = 2r - \lambda$ with d_{opt}. Here $d_{\mathrm{opt}} - d_0 = bvk/\{(v-1)(k+1)\} - (2r-\lambda) = \lambda(v-k-1)^2/(k^2-1)$. By Theorem 2.5, the code (1) in Corollary 2.3 is an optimal equidistant code if and only if $v=k+1$. When $v>k+1$, there is no possibility of getting an optimal equidistant code. But if $d_{\mathrm{opt}} - d_0 < 1$, then this code will be a nearly optimal equidistant code. Here $d_{\mathrm{opt}} - d_0 < 1$ means $\lambda(v-k-1)^2 < k^2-1$, which implies $v < k+1+\sqrt{(k^2-1)/\lambda}$. Since $k^2-2k+1 < k^2-1$, we have $k-1 < \sqrt{k^2-1}$. Therefore, when $k+1 < v \leqslant k+1+\lambda^{-1/2}(k-1)$, this code is a nearly optimal equidistant $(k+1)$-ary code in the sense of Theorem 2.5.

Theorem 2.6. *The equidistant code* (1) *constructed in Corollary* 2.3 *is optimal if and only if $v=k+1$, and is nearly optimal if $k+1 < v \leqslant k+1+\lambda^{-1/2}(k-1)$.*

Here we present two series of such nearly optimal codes.

Corollary 2.7. *Let $v \equiv 1,3 \bmod 6$, $v \geqslant 7$. Then there exists a nearly optimal equidistant code with parameters $n = v(v-1)/6$, $M = v$, $d = n-1$ and $q = v-2$.*

Proof. The parameters of the complement of $B(3,1;v)$ satisfy the inequality in Theorem 2.6. □

Corollary 2.8. *Let $v \equiv 1,4 \bmod 12$, $v \geqslant 13$. Then there exists a nearly optimal equidistant code with parameters $n = v(v-1)/12$, $M = v$, $d = n-1$ and $q = v-3$.*

Proof. The parameters of the complement of $B(4,1;v)$ satisfy the inequality in Theorem 2.6. □

Similarly we have the following.

Theorem 2.9. *The equidistant code constructed in Corollary 2.4 is nearly optimal when $s = 1$ or 2.*

3. Semiframes and q-ary codes

Inner structures of block designs are often useful for producing 'good' codes. Resolvability of a block design was used by Semakov and Zinoviev [10] to provide an optimal equidistant code. In fact, they proved that an optimal equidistant $(n,M,d;q)$ code exists if and only if there exists a resolvable BIBD with parameters $v = M$, $b = nq$, $r = n$, $k = M/q$ and $\lambda = n - d$. We apply their idea to a semiframe which is a generalization of resolvable BIBDs to get more codes.

It is known (see [6]) that in a (k,λ)-semiframe $(\mathscr{V},\mathscr{G},\mathscr{B})$ of type g^f, there are l parallel classes $\mathscr{P}$ and p partial parallel classes $\mathscr{Q}$ with respect to $G_i \in \mathscr{G}$ such that $\lambda g(f-1)/(k-1) = l + (f-1)p$.

When $l \neq 0$, i.e., there is at least one parallel class in the semiframe, k divides gf, and thus set $q = gf/k$. Choose q blocks of $\mathscr{B}$ which form a parallel class. The columns of the incidence matrix of $(\mathscr{V},\mathscr{G},\mathscr{B})$, corresponding to this parallel class, form a submatrix G, each of whose rows contains exactly one 1, and each of whose columns contains k 1's. In other words, G consists of the rows of the identity matrix I_q, each row of I_q occurring k times in G.

If $p = 0$, i.e. there is no partial parallel class (which means that the semiframe is a resolvable GDD, denoted by (k,λ)-RGDD), label the rows of I_q with $0,\dots,q-1$. Thus, we have assigned to each parallel class a q-ary column of length $gf = kq$, consisting of the symbols $0,\dots,q-1$, each of them taken k times. Therefore, we obtain a q-ary matrix with $gf = kq$ rows, $l = \lambda g(f-1)/(k-1)$ columns, and here any two rows coincide in exactly 0 or λ coordinates according as the two corresponding elements are in the same group or not, i.e., the Hamming distance between them is l or $l - \lambda$. Hence, the rows of this matrix form an $(n,M,d;q)$ code with $n = l = \lambda g(f-1)/(k-1)$, $M = gf = kq$ and $d = l - \lambda = \lambda\{g(f-1)/(k-1) - 1\}$.

Theorem 3.1. *The existence of a (k,λ)-RGDD of type g^f implies the existence of a $(\lambda g(f-1)/(k-1), gf, \lambda\{g(f-1)/(k-1)-1\};\ gf/k)$ code of constant weight $\lambda g(f-1)/(k-1)$ where for any two distinct elements x and y, $d(x,y)=\lambda g(f-1)/(k-1)$ if x and y are in the same group, and $d(x,y)=\lambda g(f-1)/(k-1)-\lambda$ otherwise.*

When $g=1$, i.e. the RGDD becomes a resolvable BIBD, denoted by RB$(k,\lambda;f)$, the Hamming distance between any two codewords is $l-\lambda$. Hence we can obtain the following result.

Corollary 3.2 (Semakov and Zinoviev [10]). *The existence of an $RB(k,\lambda;f)$ implies the existence of an optimal equidistant $(\lambda(f-1)/(k-1),\ f, \lambda\{(f-1)/(k-1)-1\};\ f/k)$ code.*

Now we consider the case $p\neq 0$, i.e., the semiframe is a so-called proper semiframe. Assign to each parallel class a q-ary column of length $gf=kq$ as described above. Similarly, we can assign to each partial parallel class a $(q-g/k+1)$-ary column of length $gf=kq$ by labelling the rows of $I_{q-g/k}$ with $1,\dots,q-g/k$ and the other rows by 0. Therefore, we obtain a q-ary matrix with $gf=kq$ rows, $l+pf$ columns, and here any two rows coincide in exactly p or λ coordinates according as the two corresponding elements are in the same group or not, i.e., the Hamming distance between them is $l+pf-p$ or $l+pf-\lambda$. Hence, the rows of this matrix form an $(n,M,d;q)$ code with $n=l+pf$, $M=gf=kq$ and $d=l+pf-\max\{p,\lambda\}$.

Theorem 3.3. *The existence of a proper (k,λ)-semiframe of type g^f implies the existence of an $(l+pf, gf, l+pf-\max\{p,\lambda\}; gf/k)$ code of constant weight $\lambda g(f-1)/(k-1)$ where for any two distinct elements x and y, $d(x,y)=l+pf-p$ if x and y are in the same group, and $d(x,y)=l+pf-\lambda$ otherwise.*

When $p=\lambda$, i.e. $l=\lambda(f-1)(g-k+1)/(k-1)$, this code is an equidistant code with distance $d=\lambda g(f-1)/(k-1)$.

Corollary 3.4. *The existence of a (k,λ)-semiframe of type g^f with $p=\lambda$ implies the existence of an equidistant $(\lambda+\lambda g(f-1)/(k-1), gf, \lambda g(f-1)/(k-1); gf/k)$ code.*

We compare $d_1=\lambda g(f-1)/(k-1)$ with $d_{\rm opt}$. Here $d_{\rm opt}-d_1=\{\lambda+\lambda g(f-1)/(k-1)\}gf(gf/k-1)/\{(gf-1)gf/k\}-\lambda g(f-1)/(k-1)=\lambda(g-k)/(fg-1)$. Thus, this equidistant code is optimal if and only if $g=k$, which means that this semiframe is a resolvable BIBD. Otherwise, we have $g>k$, which shows that there is no possibility of getting an optimal equidistant code. But if $0<\lambda(g-k)/(gf-1)<1$, we have a nearly optimal equidistant code in the sense of Theorem 2.5.

Theorem 3.5. *The equidistant code constructed in Corollary 3.4 is optimal if and only if $g=k$, and is nearly optimal whenever $\lambda<(gf-1)/(g-k)$.*

Series of such semiframes can be found in [6], which can imply the existence of the corresponding nearly optimal equidistant codes.

If $l = 0$, i.e., there is no parallel class (which means that the semiframe is a frame). Let $g(f-1)/k+1 = q$. Assign to each partial parallel class a q-ary column of length gf by labelling the rows of I_{q-1} with $1, \dots, q-1$ and other rows by 0. Then we obtain a q-ary matrix with gf rows, $pf = \lambda gf/(k-1)$ columns, and here any two rows coincide in exactly p or λ coordinates according as the two corresponding elements are in the same group or not, i.e. the Hamming distance between them is $pf - p$ or $pf - \lambda$. Hence, the rows of this matrix form an $(n, M, d; q)$ code with $n = pf$, $M = gf$ and $d = pf - \max\{p, \lambda\}$.

Theorem 3.6. *The existence of a (k,λ)-frame of type g^f implies the existence of a $(\lambda gf/(k-1), gf, \lambda gf/(k-1) - \lambda\cdot\max\{g/(k-1), 1\}; g(f-1)/k+1)$ code of constant weight $\lambda g(f-1)/(k-1)$ where for any two distinct elements x and y, $d(x,y) = \lambda g(f-1)/(k-1)$ if x and y are in the same group, and $d(x,y) = \lambda gf/(k-1) - \lambda$ otherwise.*

When $p = \lambda$, i.e. $g = k-1$, this code is an equidistant code with distance $d = \lambda(f-1)$.

Corollary 3.7. *The existence of a (k,λ)-frame of type $(k-1)^f$ implies the existence of an equidistant $(\lambda f, (k-1)f, \lambda(f-1); f-(f-1)/k)$ code.*

We compare $d_2 = \lambda(f-1)$ with d_{opt}. Here $d_{\text{opt}} - d_2 = \{\lambda f(k-1)f(f-1)(k-1)/k\}/\{\{(k-1)f-1\}\{(k-1)f+1\}/k\} - \lambda(f-1) = \lambda(f-1)/\{(k-1)^2 f^2 - 1\}$. Hence, this equidistant code can never be optimal since $\lambda \geqslant 1$ and $f > 1$, but is nearly optimal in the sense of Theorem 2.5 whenever $\lambda < \{(k-1)^2 f^2 - 1\}/(f-1)$.

Theorem 3.8. *The equidistant code constructed in Corollary 3.7 is nearly optimal whenever $\lambda < \{(k-1)^2 f^2 - 1\}/(f-1)$.*

When $g = 1$, i.e. the frame becomes an almost resolvable BIBD, denoted by ARB$(k, \lambda; f)$, the Hamming distance between any two codes is $pf - \lambda$, where $\lambda = \alpha(k-1)$ for some $\alpha \in \mathcal{N}$. Hence we can obtain the following result.

Theorem 3.9. *Let $\alpha \in \mathcal{N}$. Then the existence of an ARB$(k, \alpha(k-1); f)$ implies the existence of an equidistant $(\alpha f, f, \alpha(f-k+1); (f+k-1)/k)$ code.*

We compare $d_3 = \alpha(f-k+1)$ with d_{opt}. Here $d_{\text{opt}} - d_3 = \{\alpha f^2(f-1)/k\}/\{(f-1)(f+k-1)/k\} - \alpha(f-k+1) = \alpha(k-1)^2/(f+k-1)$. Since $k > 1$, there is no possibility of getting an optimal equidistant code. But if $0 < \alpha(k-1)^2/(f+k-1) < 1$, we have a nearly optimal equidistant code in the sense of Theorem 2.5.

Theorem 3.10. *The equidistant code constructed in Theorem 3.9 is nearly optimal whenever $\alpha < (f+k-1)/(k-1)^2$.*

Constructions and existence results of such frames and almost resolvable BIBDs can be found in [3], which can also imply the existence of the corresponding nearly optimal equidistant codes.

Finally we note that when we add $(0,\ldots,0)$ of length λf to the code constructed in Corollary 3.7, we get the code in [10].

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